

Discernment of Hubs and Clusters in Socioeconomic Networks

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Abstract

Interest in the analysis of networks has grown rapidly in the new millennium. Consequently, we promote renewed attention to a certain methodological approach introduced in 1974. Over the succeeding decade, this *two-stage-double*-standardization and hierarchical clustering (single-linkage-like)-procedure was applied to a wide variety of weighted, directed networks of a socioeconomic nature, frequently revealing the presence of “hubs”. These were, typically—in the numerous instances studied of migration flows between geographic subdivisions within nations—“cosmopolitan/non-provincial” areas, a prototypical example being the French capital, Paris. Such locations emit and absorb people broadly across their respective nations. Additionally, the two-stage procedure—which “might very well be the most successful application of cluster analysis” (R. C. Dubes, 1985)—detected many (physically or socially) isolated, functional groups (regions) of areas, such as the southern islands, Shikoku and Kyushu, of Japan, the Italian islands of Sardinia and Sicily, and the New England region of the United States. Further, we discuss a (complementary) approach developed in 1976, in which the *max-flow/min-cut* theorem was applied to *raw/non-standardized* (interindustry, as well as migration) flows.

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I. INTRODUCTION

A. L. Barabási, in his recent popular book, “Linked”, asserts that the emergence of *hubs* in networks is a surprising phenomenon that is “forbidden by both the Erdős-Rényi and Watts-Strogatz models” [1, p. 63] [2, Chap. 8]. Here, we indicate an analytical framework introduced in 1974 that the distinguished computer scientist R. C. Dubes, in a review of the compilation of multitudinous results [3], asserted “might very well be the most successful application of cluster analysis” [4, p. 142]. This *two*-stage methodology has proved insightful in revealing—among other interesting relationships—hub-like structures in networks of (weighted, directed) internodal flows. This approach, together with its many diverse socioeconomic applications, was documented in a large number of (subject-matter and technical) journal articles (among them [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]), as well as in the research institute monographs [3, 23, 24]. It has also been the subject of various comments, criticisms and discussions [21, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36] (cf. [37, 38]).

Though the principal procedure to be detailed here is applicable in a wide variety of social-science settings [3, 4], it has been primarily used, in a demographic context, to study the *internal migration* tables published at regular periodic intervals by most of the nations of the world. These tables can be thought of as $N \times N$ (square) matrices, the entries (m_{ij}) of which are the number of people who lived in geographic subdivision i at time t and j at time $t + 1$. (Some tables—but not all—have diagonal entries, m_{ii} , which may represent either the number of people who did move within area i , or simply those who lived in i both at t and $t + 1$. It can sometimes be of interest to compare analyses with zero and nonzero diagonal entries [23].)

II. TWO-STAGE METHODOLOGY

A. First Step: Double-Standardization

In the *first* step (iterative proportional fitting procedure [IPFP] [39]) of the methodology under discussion here, the rows and columns of the table of flows are alternately (biproportionally [40]) scaled to sum to a fixed number (say 1). Under broad conditions—to be discussed below—convergence occurs to a “doubly-stochastic” (bistochastic) table, with row

and column sums all simultaneously equal to 1 [41, 42, 43, 44, 45, 46]. The purpose of the scaling is to remove overall (marginal) effects of size, and focus on relative, interaction effects. Nevertheless, the *cross-product ratios* (*relative odds*), $\frac{m_{ij}m_{kl}}{m_{il}m_{kj}}$, measures of association, are left *invariant*. Additionally, the entries of the doubly-stochastic table provide *maximum entropy* estimates of the original flows, given the row and column constraints [47, 48].

For large *sparse* flow tables, only the nonzero entries, together with their row and column coordinates are needed. Row and column (biproportional) multipliers can be iteratively computed by sequentially accessing the nonzero cells [49]. If the table is “critically sparse”, various convergence difficulties may occur. Nonzero entries that are “unsupported”—that is, not part of a set of N nonzero entries, no two in the same row and column—may converge to zero and/or the biproportional multipliers may not converge [3, p. 19] [50] [51, p. 171]. The “first strongly polynomial-time algorithm for matrix scaling” was reported in [52].

The scaling was successfully implemented, in our largest analysis, with a $3,140 \times 3,140$ 1965-70 intercounty migration table—having 94.5% of its entries, zero—for the United States [9, 23], as well as for a more aggregate 510×510 table (with *State Economic Areas* as the basic unit) for the US for the same period [14]. (*Smoothing* procedures could be used to modify the zero-nonzero structure of a flow table, particularly if it is critically sparse [53, 54]. If one takes the second power of a doubly-stochastic matrix, one obtains another such matrix, but smoother in character. One might also consider standardizing the i th row [column] sum to be proportional to the number of non-zero entries in the i th row [column].)

B. Second Step: Strong Component Hierarchical Clustering

In the *second* step of the two-stage procedure, the doubly-stochastic matrix is converted to a series of *directed* (0,1) graphs (digraphs), by applying thresholds to its entries. As the thresholds are progressively lowered, larger and larger *strong components* (a directed path existing from any member of a component to any other) of the resulting graphs are found. This process (a simple variant of well-known single-linkage [nearest-neighbor or min] clustering [55]) can be represented by the familiar dendrogram or tree diagram used in hierarchical cluster analysis and cladistics/phylogeny (cf. [56, 57]).

C. Computer implementations

A FORTRAN implementation of the two-stage process was given in [58], as well as a realization in the SAS (Statistical Analysis System) framework [59]. Subsequently, the noted computer scientist R. E. Tarjan [60] devised an $O(M(\log N)^2)$ algorithm [61] and, then, a further improved $O(M(\log N))$ method [62], where N is the number of nodes and M the number of edges of a directed graph. (These substantially improved upon the earlier works [58, 59], which required the computations of *transitive closures* of graphs, and were $O(MN)$ in nature.) A FORTRAN coding—involving linked lists—of the improved Tarjan algorithm [62] was presented in [63], and applied in the aforementioned US intercounty study [23]. If the graph-theoretic (0,1)-structure of a network under study is *not* strongly connected [64], *independent* two-stage analyses of the subsystems of the network would be appropriate.

D. Goodness-of-fit

The *goodness-of-fit* of the dendrogram generated to the doubly-stochastic table itself can be evaluated—and possibly employed, it would seem, as an optimization criterion (cf. [65, p. 210] [66, sec. 3]). In the context—not of the weighted, directed networks under discussion here—but of (0,1)-networks or simply graphs, Clauset, Moore and Newman have written: “[t]he method known as *hierarchical clustering* groups vertices in networks by aggregating them iteratively in a hierarchical fashion. However, it is not clear that the hierarchical structures produced by these and other popular methods are unbiased, as is also the case for the hierarchical clustering algorithms of machine learning. That is, it is not clear to what degree these structures reflect the true structure of the network, and to what degree they are artifacts of the algorithm itself. This conflation of intrinsic network properties with features of the algorithms used to infer them is unfortunate . . . we give a precise definition of hierarchical structure, give a generic model for generating arbitrary hierarchical structure in a random graph, and describe a statistically principled way to learn the set of hierarchical features that most plausibly explain a particular real-world network”. [66].

Distances between nodes in the dendrogram satisfy the (stronger than *triangular*) *ultra-metric* inequality, $d_{ij} \leq \max(d_{ik}, d_{jk})$ [67, p. 245] [68, eq. (2.2)].

III. EMPIRICAL RESULTS

A. Cosmopolitan or Hub-Like Units

1. Internal migration flows

Geographic subdivisions (or groups of subdivisions) that enter into the bulk of the dendrogram at the weakest levels are those with the broadest ties. Typically, these have been found to be “cosmopolitan”, hub-like areas, a prototypical example being the French capital, Paris [3, sec. 4.1] [6]. Similarly, in parallel analyses of other internal migration tables, the cosmopolitan/non-provincial natures of London [69], Barcelona [16] [3, sec. 6.2, Figs. 36, 37], Milan [12] [3, sec. 6.3, Figs. 39, 40] (cf. [13]), Amsterdam [3, p. 78] [25], West Berlin [3, p. 80], Moscow (the city and the oblast as a unit) [19] [3, sec. 5.1 and Figs. 6, 7], Manila (coupled with suburban Rizal) [70], Bucharest [18], Île-de-Montréal [3, p. 87], Zürich, Santiago, Tunis and Istanbul [71] were—among others—highlighted in the respective dendrograms for their nations [3, sec. 8.2] [15, pp. 181-182] [8, p. 55]. In the intercounty analysis for the US, the most cosmopolitan entities were: (1) the *centrally* located paired Illinois counties of Cook (Chicago) and neighboring, suburban Du Page; (2) the nation’s capital, Washington, D. C.; and (3) the paired south Florida (retirement) counties of Dade (Miami) and Broward (Ft. Lauderdale) [9, 23, 72]. In general, counties with large military installations, large college populations or state capitals also interacted broadly with other areas [23, p. 153]. Application of the two-stage methodology to 1965-66 London inter-borough migration [25] indicated that the three inner boroughs of Kensington and Chelsea, Westminster, and Hammersmith acted—as a unit—in a cosmopolitan manner [3, sec. 5.2, Fig. 10]. (In sec. 8.2 and Table 16 of the anthology of results [3], additional geographic units and groups of units found to be cosmopolitan with regard to migration, are enumerated.)

It should be emphasized that although the indicated cosmopolitan areas may generally have relatively large populations, this can not, in and of itself, explain the wide national ties observed, since the double-standardization, in effect, renders all areas of equal overall size. (However, to the extent that larger areas do have fewer zero entries in their corresponding rows and columns, a bias to cosmopolitanism may in fact be present, which should be carefully considered. Possible corrections for bias were discussed above in sec. II A.) If one were to obtain a (zero-diagonal) doubly-stochastic matrix, all the entries of which were simply $\frac{1}{N-1}$,

it would indicate complete indifference among migrants as to where they come from and to where they go. A maximally cosmopolitan unit would be one for which all the corresponding row and column entries were $\frac{1}{N-1}$ (if all the diagonal entries, m_{ii} , are *a priori* zero). (It seems interesting to note that cosmopolitan areas appear to have a certain *minimax* character, that is, the maximum doubly-stochastic entry for the corresponding row and column tends to be minimized.)

2. Trade and interindustry flows

The nation of Italy possessed the broadest ties in a two-stage analysis of the value of 1974 trade between 113 nations, followed by a closely-bound group composed of the four Scandinavian countries [17] [3, sec. 5.6, Fig. 22]. In a two-stage study (but using *weak* rather than strong components of the associated digraphs) of the 1967 US interindustry transaction table, the industry with the broadest (most diffuse) ties was found to be Other Fabricated Metal Products [10, 73] [24, pp. 13-18].

3. Journal citations

In a two-stage analysis of 22 mathematical journals, the *Annals of Mathematics* and *Inventiones Mathematicae* were strongly paired, while the *Proceedings of the American Mathematical Society* was found to possess the broadest, most diffuse ties [8].

In a recent, large-scale ($N > 6000$) journal-to-journal citation analysis, decomposing “the network into modules by compressing a description of the probability flow”, Rosvall and Bergstrom preliminarily *omitted* from their analysis the prominent journals *Science*, *Nature* and *Proceedings of the National Academy of Sciences* [74, p. 1123]. (Those are precisely the ones that would be expected to be “cosmopolitan” or hub-like in character, and to be highlighted in a corresponding two-stage analysis.) Their rationale for the omission was that “the broad scope of these journals otherwise creates an illusion of tighter connections among disciplines, when in fact few readers of the physics articles in *Science* also are close readers of the biomedical articles therein”. (In [24, pp. 125-153], we reported the results of a *partial* hierarchical clustering—not a two-stage analysis, but one originally designed and conducted by Henry G. Small and William Shaw—of citations between more than 3,000 journals. The

clusters obtained there were compared with the actual subject matter classification employed by the Institute for Scientific Information.)

B. Functional Clusters of Units

1. Internal migration regions

Geographically isolated (insular) areas—such as the Japanese islands of Kyushu and Shikoku [5]—emerged as well-defined *clusters* (regions) of their constituent (seven and four, respectively) subdivisions (“prefectures” in the Japanese case) in the dendrograms for the two-stage analyses, and similarly the Italian islands of Sicily and Sardinia [12], the North and South Islands of New Zealand, and the Canadian islands of Newfoundland and Prince Edward Island [3, p. 90] (cf. [75, 76]). The eight counties of Connecticut, and other New England groupings, as further examples, were also very prominent in the highly disaggregated US analysis [23]. Relatedly, in a study based solely upon the 1968 movement of *college students* among the fifty states, the six New England states were strongly clustered [11, Fig. 1]. Employing a 1963 Spanish interprovincial migration table, well-defined regions were formed by the two provinces of the Canary Islands, and the four provinces of Galicia [16] [3, sec. 6.2.1, Fig. 37]. The southernmost Indian states of Kerala and Madras (now Tamil Nadu) were strongly paired on the basis of 1961 interstate flows [22]. A detailed comparison between functional migration regions found by the two-stage procedure and those actually employed for administrative, political purposes in the corresponding nations is given in sec. 8.1 and Table 15 of [3].

It should be noted that it is rare that the two-stage methodology yields a migration region composed of two or more noncontiguous subregions—even though no contiguity information at all is present in the flow table nor provided to the algorithm (cf. [54, 77]). A notable exception to this rule was the uniting of the northern Italian region of Piemonte—the location of industrial Turin, where Fiat is based—with southern regions, *before* joining with central regions, in an 18-region 1955-70 study [13] [3, p. 75] (cf. [12]).

2. Intermarriage and interindustry clusters

In a two-stage analysis of a 32×32 table of birthplace of bridegroom versus birthplace of bride of 1947 Australian intermarriages [78], Greece and Cyprus were the strongest dyad [3, sec. 5.7, Fig. 25].

In the 1967 US interindustry two-stage (*weak* component) analysis, two particularly salient pairs of functionally-linked industries were: (1) Stone and Clay Products, and Stone and Clay Mining and Quarrying; and (2) Household Appliances and Service Industry Machines (the latter industry purchases laundry equipment, refrigerators and freezers from the former) [10, 73] [24, pp. 13-18].

IV. STATISTICAL ASPECTS

It would be of interest to develop a theory—making use of the rich mathematical structure of doubly-stochastic matrices—by which the *statistical significance* of apparent hubs and clusters in dendrograms produced by the two-stage procedure could be evaluated [23, pp. 7-8] [79]. In the geographic context of internal migration tables, where nearby areas have a strong distance-adversion predilection for binding, it seems unlikely that most clustering results generated could be considered to be—in any standard sense—“random” in nature. On the other hand, other types of “origin-destination” tables, such as those for *occupational* mobility [80], journal citations [8] [24, pp. 125-153], interindustry (input-output) flows [10] [73], brand-switches [3, sec. 9.6] [81], crime-switches [3, sec. 9.7] [82, Table XII], and (Morse code) confusions [3, sec. 9.8] [83], among others, clearly lack such a geographic dimension (cf. [84]). An efficient algorithm—considered as a nonlinear dynamical system—to generate *random* bistochastic matrices has recently been presented [43] (cf. [85, 86]).

In the US 3,140-county migration study, a statistical test of Ling [87] (designed for *undirected* graphs), based on the difference in the ranks of two edges, was employed in a heuristic manner [23, pp. 7-8]. For example, the 3,148th largest doubly-stochastic value, 0.12972 (corresponding to the flow from Maui County to Hawaii County), *united* the four counties of the state of Hawaii. The (considerably weaker) 7,939th largest value, 0.07340 (the link from Kauai County, Hawaii, to Nome, Alaska), *integrated* the four-county state of Hawaii into a much larger 2,464-county cluster. The difference of these two ranks, $4,192 = 7,340 - 3,148$,

is the isolation index or “survival time” of this state as a cluster. Reference to Table 1 in [23] showed the significance of the state of Hawaii as a functional internal migration unit at the 0.01 level [23, p. 7]. (In the computation of this table, the approximation was used that the number of edges in the relevant digraphs was a negligible proportion of all possible $3,140 \times 3,139$ edges.)

Also, the possibility of employing the *asymptotic* theory of random digraphs [88, 89] for statistical testing purposes was raised in [23]. In this regard, it was necessary to consider the 38,815 largest entry of the doubly-stochastic matrix to complete the hierarchical clustering of the 3,140 counties. The probability is 0.973469 that a random digraph with 3,140 nodes and 38,414 links is strongly connected [89, p. 361], where $0.973469 = e^{-2e^{-4.30917}}$, and $38,814 = 3140(\log 3140 + 4.30917)$. Evidence of systematic structure in the migration flows can, thus, be adduced, since the digraph based on the 38,814 greatest-valued links was *not* strongly connected [23, p. 8] (cf. [90]).

In a random digraph with a large number of nodes, the probability is close to one that all nodes are either isolated or lie in a single (“giant”) strong component. The existence of intermediate-sized clusters is thus evidence of non-randomness, even if such groups are not themselves significant according to the isolation (difference-of-ranks) criterion of Ling [87]. With randomly-generated data and many taxonomic units, one would expect the two-stage procedure to yield a dendrogram exhibiting complete chaining. So, although single-linkage clustering is often criticized for producing chaining, chains can also be viewed simply as indications of inherent randomness in the data. In contrast to single-linkage clustering, strong component hierarchical clustering can merge *more* than two clusters (children) into one (parent) node. This serves to explain why fewer clusters (2,245) were generated in the intercounty migration study than the 3,139 that single-linkage (in the absence of ties) would produce.

A. A cluster-analytic isolation criterion

Dubes and Jain [91] provided “a semi-tutorial review of the state-of-the-art in cluster validity, or the verification of results from clustering algorithms”. Among other evaluative standards, they discussed isolation criteria, which “measure the distinctiveness or separation or gaps between a cluster and its environment”. Such a statistic was developed and applied

in [92] in order to extract a small proportion of 5,385 clusters (3,140 of them single units, 673 pairs, 230 triples, 104 quartets,...) for detailed examination based on the two-stage analysis of the 1965-1970 United States intercounty migration table [23].

The largest value of the isolation criterion, for all clusters of fewer than 2940 units, was attained by a region formed by the eight constituent counties of the state of Connecticut. (Groups formed by the application of the two-stage procedure to interareal migration data are, as a strong rule, composed of contiguous areas [3, 15]. This occurs even in the absence of contiguity constraints, reflecting the distance decay of migration.) The 11,080th largest doubly-standardized entry, 5,666, corresponding to movement from New Haven to (New York City suburban) Fairfield, unified these eight counties (all row and column sums had been adjusted to 100,000). Not until the 16,047th largest doubly-standardized value, 4,085 (the functional linkage from Litchfield, Connecticut to Berkshire, Massachusetts), viewing the clustering procedure as an agglomerative one, was Connecticut absorbed into a larger region. The isolation criterion for Connecticut is set equal to

$$25.3175 = -\log \left[\left((8 \times 7 + 3132 \times 3131) / (3140 \times 3139) \right)^{(16047-11080)} \right] \quad (1)$$

The term in large parentheses is the proportion of cells in the $3,140 \times 3,140$ table associated with either movement within Connecticut or within the set of 3,132 complementary counties (since intracounty flows are not available, a diagonal correction is made). This term, raised to the power shown, is the probability (unadjusted for occupied cells) that *none* of $4967 = 16047 - 11080$ consecutive doubly-standardized values would correspond to movement between Connecticut and its complement. Such a Connecticut-complement linkage could possibly result in a merger: an *unobserved* phenomenon. (For further details, including maps, discussion and extensive applications of the isolation criterion developed to the U. S. intercounty analysis, see [92].) This isolation score for the cluster formed by the four counties of Hawaii—discussed above—was 12.21, while the District of Columbia had the highest score, 23.81, for any single county [92, Table I].

V. COMPLEMENTARY NETWORK FLOW PROCEDURE

The creative, productive network analyst M. E. J. Newman has written: “Edge weights in networks have, with some exceptions ... received relatively little attention in the *physics*

[emphasis added] literature for the excellent reason that in any field one is well advised to look at the simple cases first (unweighted networks). On the other hand, there are many cases where edge weights are known for networks, and to ignore them is to throw out a lot of data that, in theory at least, could help us to understand these systems better” [93]. Of course, the numerous (mostly, internal migration) applications of the two-stage procedure we have discussed above have, in fact, been to such weighted (and directed) networks.

In [93], Newman applied the famous Ford-Fulkerson *max-flow/min-cut theorem* [94, Chap. 22] to weighted networks (which he mapped onto unweighted *multigraphs*). Earlier, this theorem had been used to study Spanish [76], Philippine [95], and Brazilian, Mexican and Argentinian [96] internal migration, US interindustry flows [24, pp. 18-28] [97] [73, sec. III] and the international flow of college students [21] (cf. [98])—all the corresponding flows now being left unadjusted, that is *not* (doubly- nor singly-) standardized.

In this “multiterminal” approach, the maximum flow and the dual minimum edge cut-sets, between *all* ordered pairs of nodes are found. Those cuts (often few or even *null* in number) which partition the N nodes nontrivially—that is, into two sets each of cardinality greater than 1—are noted. The set in each such pair with the fewer nodes is regarded as a nodal cluster (region, in the geographic context). It has the interesting, defining property that fewer people migrate into (from) it, as a whole, than into (from) its node. In the Spanish context, the (nodal) province of Badajoz was found to have a particularly large out-migration sphere of influence, and the (Basque) province of Vizcaya (site of Bilbao and Guernica), an extensive in-migration field [76]. In an analysis of 1967 US interindustry transactions based on 468 industries, among the industries functioning as nodes of *production* complexes with large numbers of members were: Advertising; Blast Furnaces and Steel Mills; Electronic Components; and Paperboard Containers and Boxes. Conversely, among those serving as nodes of *consumption* complexes were Petroleum Refining and Meat Animals [73, 97].

VI. CONCLUDING REMARKS

The networks formed by the World Wide Web and the Internet have been the focus of much recent interest [1]. Their structures are typically represented by $N \times N$ *adjacency* matrices, the entries of which are simply 0 or 1, rather than nonnegative numbers, as in internal migration and other flow tables. One might investigate whether the two-stage

double-standardization and hierarchical clustering, and the (complementary) multiterminal max-flow/min-cut procedures we have sought to bring to the attention of the active body of contemporary network theorists, could yield novel insights into these and other important modern structures.

Though quite successful, evidently, in simultaneously revealing *both* hub-like and clustering behavior in recorded flows, the indicated implementations of the two-stage procedure did not address the recently-emerging, theoretically-important issues of scale-free networks, power-law descriptions, network evolution and vulnerability, and small-world properties, among others, that have been stressed by Barabási [1] (and his colleagues and many others in the growing field [99]). (For critiques of these matters, see [100, 101].) One might—using the indicated two-stage procedure—compare the hierarchical structure of geographic areas using internal migration tables at *different* levels of geographic aggregation (counties, states, regions...) (cf. [84]). To again use the example of France, based on a 1962-68 21×21 inter-regional table, Région Parisienne was the most hub-like [3, sec. 4.1] [6], while using a finer 89×89 1954-62 interdepartmental table, the dyad composed of Seine (that is Paris and its immediate suburbs) together with the encircling Seine-et-Oise (administratively eliminated in 1964) was most cosmopolitan [7] [3, sec. 6.1]. (In [84], “two distinct approaches to assessing the effect of geographic scale on spatial interactions” were developed.)

VII. AFTERWORD

It might be of interest to describe the immediate motivation for this particular communication. I had done no further work applying the methods described above after 1986, being aware of, but not absorbed in recent developments in network analysis. In May, 2008, Mathematical Reviews asked me to review the book of Tom Siegfried [2], chapter 8 of which is devoted to the on-going activities in network analysis. This further led me (thanks to D. E. Boyce) to the book of Barabási [1]. I, then, e-mailed Barabási, pointing out the use of the clustering methodologies described above. In reply, he wrote, in part: “I guess you were another demo of everything being a question of timing—after a quick look it does appear that many things you did have come back as questions – with much more detailed data—again in the network community today. No, I was not aware of your papers, unfortunately, and it is hard to know how to get them back into the flow of the system.” The present com-

munication might be seen as an effort in that direction, alerting present-day investigators to these demonstratedly fruitful research methodologies, and suggesting possible further applications and theoretical analysis. (Additionally, we sent Barabási the two-stage analysis [18] for a 1972 40×40 interdistrict migration table for his native country, Romania—in which the capital of Bucharest was featured as most cosmopolitan in nature, and the coupled Black Sea districts of Constanța and Tulcea, as next most. His reply was: “Cool-thanks”.)

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